Exact time-average distribution for a stationary non-Markovian massive Brownian particle coupled to two heat baths

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Using a time-averaging technique we obtain exactly the probability distribution for position and velocity of a Brownian particle under the influence of two heat baths at different temperatures. These baths are expressed by a white noise term, representing the fast dynamics, and a colored noise term, representing the slow dynamics. Our exact solution scheme accounts for inertial effects, that are not present in approaches that assume the Brownian particle in the over−damped limit. We are also able to obtain the contribution associated with the fast noise that are usually neglected by other approaches.

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I. INTRODUCTION

The role of time-scales is crucial for understanding the validity of equilibrium techniques to any physical process since fast degrees of freedom (time-scale τ_f) generally reach their stationary distribution well within the experiment's duration but the same cannot be assumed about slow ones (time-scale τ_s). For instance, glasses are a very important class of systems presenting slow dynamics [1].

The probability distribution for the slow degrees of freedom depends on the system's preparation and also on the elapsed time interval, i.e., for how long the system has aged [1]. For the time interval $\tau_f \ll t \ll \tau_s$, the system behaves as driven by an effective external slow field associated with the slow variables. However, there is no physical reason for the slow and fast fluctuations distribution to match, so, in general, $T_{slow} \neq T_{fast}$. In this case, the reduced form for the fast variable's distribution is an instantaneous equilibrium one [2]. For longer times, $t \gg \tau_s$, the distributions for all variables (fast and slow) reach a stationary state. For closed systems governed by a microscopic Hamiltonian, the stationary state will be a true equilibrium one. For open systems in contact with a single thermal reservoir at temperature T, the final state will also be the equilibrium distribution. However, by subjecting a system to simultaneous contacts with two (or more) reservoirs at distinct temperatures, say T_1 and T_2, it will reach a stationary state distinct from the Boltzmann-Gibbs (BG) equilibrium [3]. In consequence, the stationary state will depend on the properties of system-reservoir interactions and the reservoirs temperatures. Simple models presenting these characteristics, such as Brownian Particles (BP), have been used to explore the physics of slow dynamics and glasses [4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

The complete description of BP is given by its microscopic interactions with a heat bath of lighter particles as well as the interactions with the external environment. Trying to evaluate the macroscopic behavior starting up from a microscopic model presents an impossible task. However, for many physical systems we can often reduce the number of important variables down to a manageable set, thanks to well separated time-scales [14, 15, 16] allowing us to eliminate the fast variables of the problem. The effect of the eliminated variables is taken into account by means of a random forcing term on the equations of motion for the remaining variables [2, 17] and by suitable friction coefficients. The random term (noise function) represents the effect of the collisions of the BP with the heat baths's particles. In the case these thermal forces vary on a very short time-scale, their effect can be represented by uncorrelated collisions (white noise) and the BP motion can be analyzed in a simple way because its stochastic dynamics can be treated as a Markovian process. Finally, its long term behavior is well described by a Boltzmann-Gibbs (BG) equilibrium [18]. However, for many realistic Brownian processes in dense fluids the short time-scale approximation for the random forces may not be accurate, since the time-intervals of the microscopic collisions might overlap. Instead,

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we need to consider that those forces act upon the BP during a well defined time-scale, or colored noise. Also, the presence of excited large-scale hydrodynamic modes will affect the BP by long time-scale feed-back loops [19, 20] that can be accounted for by a dissipative memory function term in the Langevin-like equation describing the Brownian motion [18].

Our present goal is to obtain the long-time stationary state for a BP system, subjected to contacts with two different heat baths at distinct temperatures, that is simple enough to be treated exactly but sophisticated enough to present the non-trivial stationary properties discussed above [21, 22].

In our model, it is possible to keep the BP from reaching the BG equilibrium by simultaneously subjecting it to the influence of two thermal baths at different temperatures T_1 and T_2, one acts as heat source while the other acts as a heat sink. As for the BP simultaneously subjected to T_1 and T_2, the stationary state will depend on the details of the interaction between the baths and the BP and the effective temperature will be an intermediary one between T_1 and T_2, as will be shown. Even if temperature gradients are not present, the stationary distribution for velocities and positions will differ from the BG measure, reducing to it only when the temperatures are the same, $T_1 = T_2 = T$. This simple model has been proposed to describe the behavior of glasses [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 23, 24] where one assumes different time-scales for each thermal bath, via their noise properties: the short time-scale white noise reproduces thermal relaxation whereas the long time-scale noise reproduces very slow, correlated, structural rearrangement of the glass. Additionally, in our approach we can include the effects related to the finite mass of the particle in stationary state, as well as other effects associated with the slow noise [3] which are often neglected by other methods based on the over-damped limit calculations (which corresponds to the limit $m \to 0$). For instance, the BP adjusts itself to the slow noise function as [3, 23], specially when its mass can be neglected (over-damped case), and the position distribution becomes very similar to a true equilibrium form, given that the slow noise acts like an external effective potential [3]. However, this simplification eliminates the contribution of the fast noise for the stationary distribution [3]. In contrast, our time-averaging scheme will integrate that small effect for a very long time and exhibit the missing contribution.

This paper is organized as follows. In Section II we describe the model. In Section III we calculate the time-averaging integrals obtaining the displacement and velocity non-vanishing terms. In Sections IV and V, we obtain the stationary (or equilibrium) distribution and discuss our main results.

II. EXACTLY SOLVABLE MODEL

A. Langevin-like Equation

Let's consider a massive Brownian Particle (BP) moving under the action of a confining potential $V(x)$, which is in contact with two heat baths of distinct temperatures and time-scales, expressed by the noise functions $\xi(t)$ and $\eta(t)$. There are some interesting physical systems that present this kind of multiple time-scale phenomena. For instance, in glasses the short time-scale (fast) modes are associated with thermal vibrations and the long time-scale (slow) ones with structural reorganization of the atomic structure [23].

The effect of the fast modes on the BP are represented by the white noise term $\eta(t)$, associated with the heat bath at temperature T_1 and the instantaneous friction coefficient Γ_1. On the other hand, the slow modes contribution are represented by a colored noise term $\xi(t)$, associated with the heat bath at temperature T_2 (in general $T_2 \neq T_1$). These modes give rise to long-time feed-back loops that are expressed by the integration of a frictional memory function $\phi(t-t^{'})$ over time.

This is modeled by two coupled stochastic differential equations:

$$\dot{x} = v, \tag{1}$$

$$m\,\dot{v} = -\frac{\partial V}{\partial x} - \int_0^t dt^{'}\,\phi(t-t^{'})v(t^{'}) - \Gamma_1 v(t) + \xi(t) + \eta(t), \tag{2}$$

where the properties of the terms above are discussed in the following. We assume $V(x) = \frac{1}{2}kx^2$ for simplicity.

B. Noise properties

The stochastic process described by Eq.(2) is non-Markovian due to the noise ξ be time correlated with a well defined time-scale τ and exponential time-correlation function, or a colored noise (see for example [25]). It represents

the slow dynamics and only at time-scales much larger than τ will ξ manifest Markovian behavior. The noise η is a white one and represents the fast dynamics. Both noises, fast and slow, are Gaussian and can be defined in terms of their two lowest cumulants. Let's start with the slow noise properties:

$$\langle \xi(t) \rangle = 0, \tag{3}$$

$$\langle \xi(t)\xi(t^{'}) \rangle = \frac{D_2}{\tau} \exp\left(-\frac{|t-t^{'}|}{\tau} \right) = \frac{\Gamma_2\, T_2}{\tau} \exp\left(-\frac{|t-t^{'}|}{\tau} \right), \tag{4}$$

giving the memory function [26]

$$\phi(t-t^{'}) = \frac{\langle \xi(t)\xi(t^{'}) \rangle}{T_2} = \frac{\Gamma_2}{\tau} \exp\left(-\frac{|t-t^{'}|}{\tau} \right), \tag{5}$$

where τ is the characteristic time-scale of the slow noise, T_2 its temperature and Γ_2 a dissipative strength associated with ξ. The units are chosen so that Boltzmann's constant is equal to one ($k_B = 1$).

The first two cumulants for the fast noise are given below:

$$\langle \eta(t) \rangle = 0, \tag{6}$$

$$\langle \eta(t)\eta(t^{'}) \rangle = T_1\, \Gamma_1\, \delta(t-t'), \tag{7}$$

where T_1 is the temperature and Γ_1 is the dissipative coefficient for the fast noise.

C. Equilibrium distribution

Our goal is to find out the exact stationary solution for Eq.(2) through the time-averaging of

$$\rho(x,v,t) = \delta(x - x(x_o, v_o, t))\delta(v - v(x_o, v_o, t)),$$

as $t \to \infty$. The functions $x(x_o, v_o, t))$ and $v(x_o, v_o, t)$ are the solutions for a given set of BP's initial conditions (x_o, v_o) and for a given realization of the stochastic processes $\xi(t)$ and $\eta(t)$ [18]. We need to take the average over these last two stochastic processes in order to obtain the stationary solution.

Thus, the stationary distribution for the Brownian degrees of freedom reads [26]:

$$\begin{aligned} P^{ss}(x,v) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\ \langle \rho(x,v,t) \rangle_{\xi,\eta} \\ &= \lim_{z\to 0^+} z \int_0^\infty dt\, e^{-zt} \langle \rho(x,v,t) \rangle_{\xi,\eta} \end{aligned} \tag{8}$$

For simplicity, we will assume that the initial conditions are $x_o = v_o = 0$, since the terms depending on the initial conditions tend to zero as $t \to \infty$ [26].

D. Laplace transformations

The centerpiece of our method is to use Laplace transforms in order to make our problem manageable and the mathematical reason for doing so is because Eq.(8) itself is the Laplace transformation of the averaged density. Thus, our goal is to describe the stationary state probability distribution, $P^{ss}(x,v)$, in terms of the Laplace transform of the noise functions, $\tilde{\xi}(z)$ and $\tilde{\eta}(z)$ [26].

In fact, we can express $P^{ss}(x,v)$ as a sum of many terms involving $\tilde{\xi}(z)$ and $\tilde{\eta}(z)$. Most of these terms will not contribute to the distribution, but we can calculate analytically the remaining ones, finding the exact expression for $P^{ss}(x,v)$. In order to do that we need to express the Laplace transforms, of the position and velocity of the BP, as functions of $\tilde{\xi}(z)$ and $\tilde{\eta}(z)$. In the following we proceed to implement the strategy just described.

We start by taking the Laplace transformations of Eqs.(1) and (2) (with $\text{Re}(z) > 0$):

$$\left[m\,z + \frac{\Gamma_2}{\tau\,z + 1} + \Gamma_1 + \frac{k}{z}\right]\,\tilde{v}(z) = \tilde{\xi}(z) + \tilde{\eta}(z), \tag{9}$$

and

$$z\,\tilde{x}(z) = \tilde{v}(z). \tag{10}$$

The Laplace transforms for the noise correlations are given by [33]:

$$\begin{aligned}\langle\tilde{\xi}(iq_i + \epsilon)\,\tilde{\xi}(iq_j + \epsilon)\rangle \;&=\; \left\{\frac{2\,\Gamma_2\,T_2}{[i(q_i + q_j) + 2\epsilon]\,[1 - \tau(iq_i + \epsilon)]\,[1 - \tau(iq_j + \epsilon)]}\right\} + \\ &-\;\Gamma_2\,T_2\,\tau\left\{\frac{3 + \tau\,[i(q_i + q_j) + 2\epsilon] - \tau^2(iq_i + \epsilon)(iq_j + \epsilon)}{[1 - \tau(iq_i + \epsilon)]\,[1 + \tau(iq_i + \epsilon)]\,[1 - \tau(iq_j + \epsilon)]\,[1 + \tau(iq_j + \epsilon)]}\right\},\end{aligned} \tag{11}$$

$$\langle\tilde{\eta}(iq_i + \epsilon)\,\tilde{\eta}(iq_j + \epsilon)\rangle = \frac{2\,\Gamma_1\,T_1}{i(q_i + q_j) + 2\epsilon}, \tag{12}$$

$$\langle\tilde{\eta}(iq_i + \epsilon)\,\tilde{\xi}(iq_i + \epsilon)\rangle = 0. \tag{13}$$

It can be shown that the second member of the RHS of Eq.(11) does not contribute to the stationary distribution $P^{ss}(x, v)$ when $z \to 0$ in Eq.(8). The above Eqs.(11), (12) and (13) will be crucial for calculating $P^{ss}(x, v)$ via Eq.(8).

We observe that when $\Gamma_2 \to 0$ we recover previous results [26] since the contribution of the slow noise disappears. Furthermore, we notice that by taking the limit $\tau \to 0$ the colored noise ξ becomes a white one, which is reflected in the fact that its contribution in Eqs.(2) and (9) becomes similar to the one by η.

Following a strategy similar to the used in Ref. [26], we express the Laplace transformation of position and velocity as functions of the Laplace transformation of the noise functions:

$$\tilde{v}(z) = \Theta(z)\left[\tilde{\xi}(z) + \tilde{\eta}(z)\right], \tag{14}$$

$$\tilde{x}(z) = \Omega(z)\left[\tilde{\xi}(z) + \tilde{\eta}(z)\right], \tag{15}$$

with

$$\Theta(z) = z\,\Omega(z) = \frac{z\,(\tau\,z + 1)}{m\,\tau\left(z^3 + \frac{\theta}{m\,\tau}\,z^2 + \frac{\omega}{m\,\tau}\,z + \frac{k}{m\,\tau}\right)}, \tag{16}$$

where θ and ω are given by

$$\theta \equiv m + \Gamma_1\,\tau, \tag{17}$$

and

$$\omega \equiv \Gamma_1 + \Gamma_2 + k\,\tau. \tag{18}$$

For simplicity, we assumed the initial conditions $x(t = 0) = 0$ and $v(t = 0) = 0$, since for $T \to \infty$ the terms carrying the memory of the initial conditions will decay to zero [26]. The denominator of $\Theta(z)$ and $\Omega(z)$ is of the third order on z, thus, there are three distinct roots for:

$$z^3 + \frac{\theta}{m\,\tau}\,z^2 + \frac{\omega}{m\,\tau}\,z + \frac{k}{m\,\tau} = 0, \tag{19}$$

namely z_1, z_2 and z_3:

$$z_1 = -\frac{(\theta\,\lambda + \sigma - \lambda^2)}{3\,m\,\tau\,\lambda}, \tag{20}$$

$$z_2 = z_3^* = -\frac{(2\,\theta\,\lambda - \sigma + \lambda^2)}{6\,m\,\tau\,\lambda} + i\frac{\sqrt{3}\,(\sigma + \lambda^2)}{6\,m\,\tau\,\lambda}, \tag{21}$$

where

$$\sigma \equiv -\theta^2 + 3\,\omega\,m\,\tau, \tag{22}$$

and

$$\lambda \equiv \frac{1}{2^{1/3}}\sqrt[3]{9\,\omega\,\theta\,m\tau - 27\,k\,m\tau^2 - 2\,\theta^3 + 3\,m\tau\,[3\,(4\,\omega^3 m\tau - \omega^2\theta^2 - 18\,\omega\,\theta\,m\tau\,k + 27\,k^2 m\tau^2 + 4\,k\,\theta^3)]^{1/2}}. \tag{23}$$

We define the quantities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{F}$ and $\mathcal{G}$ as:

$$z_2 \equiv \mathcal{A} + i\,\mathcal{B}\,;\; z_1 - z_2 \equiv \mathcal{C} + i\,\mathcal{D}\,;\; z_1 + z_2 \equiv \mathcal{F} + i\,\mathcal{G}. \tag{24}$$

The terms $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{F}$ and $\mathcal{G}$ are important because we perform residue integrations over their poles [26]. It is important to compute the expressions for the leading terms in the expressions (22), (23) and (24) when $m\,\tau \to 0$, i.e., in the over−damped limit. Since $\theta > 0$, in linear order on $m\,\tau$ we have

$$\lambda \;=\; (-\theta)\left(1 - \frac{3\,\omega\,m\,\tau}{2\,\theta^2} - \frac{m\,\tau\,\sqrt{3}}{2\,\theta^2}\sqrt{-\omega^2 + 4\,k\,\theta}\right). \tag{25}$$

The function σ is already linear in $m\,\tau$. In the leading order in $m\,\tau$, we obtain:

$$\mathcal{A} \sim -\frac{\omega}{2\,\theta},\, \mathcal{B} \sim \frac{\sqrt{-\omega^2 + 4\,\theta\,k}}{2\,\theta},\, \mathcal{C} \sim -\frac{\theta}{m\,\tau},\, \mathcal{D} \sim -\frac{\sqrt{-\omega^2 + 4\,\theta\,k}}{2\,\theta},\, \mathcal{F} \sim -\frac{\theta}{m\,\tau},\, \mathcal{G} \sim \frac{\sqrt{-\omega^2 + 4\,\theta\,k}}{2\,\theta}. \tag{26}$$

E. Calculating the stationary distribution

In Appendix A we show the derivation for the following equation (originally obtained in Ref. [26]) for the stationary state distribution Eq. 8:

$$\begin{aligned} P^{ss}(x,v) \;&=\; \lim_{z,\epsilon\to 0^+}\sum_{l,m=0}^{\infty}\int_{-\infty}^{+\infty}\frac{dQ}{2\pi}\frac{dP}{2\pi}e^{iQx+iPv}\frac{(-iQ)^l}{l!}\frac{(-iP)^m}{m!}\int_{-\infty}^{+\infty}\prod_{f=1}^{l}\frac{dq_f}{2\pi}\prod_{h=1}^{m}\frac{dp_h}{2\pi} \\ &\times\; \frac{z}{z - \left[\sum_{f=1}^{l} iq_f + \sum_{h=1}^{m} ip_h + (l+m)\epsilon\right]}\langle\prod_{f=1}^{l}\tilde{x}(iq_f+\epsilon)\prod_{h=1}^{m}\tilde{v}(ip_h+\epsilon)\rangle \end{aligned} \tag{27}$$

In order to compute P^{ss} we need to replace Eqs.(11) to (15), into Eq.(27). The integrations paths for the variables $\{q_f, p_h\}$ can be chosen to be identical to the one shown in Fig.(1): the integrations in Eq.(27) will correspond to a series of residue integrations over (only some) of the multiple poles of its integrand. These poles correspond to the ones associated with the noise functions (Eqs.(11) to (15)) and the ones given by Eq.(16) and the function $I(z)$:

$$I(z) = \frac{z}{z - \left[\sum_{f=1}^{l} iq_f + \sum_{h=1}^{m} ip_h + (l+m)\epsilon\right]}. \tag{28}$$

In special, the function $I(z)$ is crucial for our computations but its role only becomes clear near the end of the calculations. As a matter of fact, the pole of I(z) is located outside of the integration path for all $\{q_f, p_h\}$, as in Fig.(1). The really important poles are associated with the roots of the denominators of $\Theta(z)$ and $\Omega(z)$, z_1, z_2 and z_3 above (the z-poles), and the ones from the denominators of the Laplace transforms for the noise variances (the noise-poles), Eqs.(11) and (12), that are locate on the inside of the path of Fig.(1). After replacing the rather complicated forms for the noise Laplace transforms into Eq.(27), the computing of $P^{ss}(x,v)$ becomes a task of doing the residue integrations around the relevant poles and collecting the non-vanishing terms.

The effect on $I(z)$ due to the residue integration of Eq.(27) over its relevant poles is rather interesting. The integrations over the noise-poles for $\{q_f, p_h\}$ that are present in $I(z)$ imply that the value the denominator of $I(z)$ will take has a non-zero real part, even if only one of such an integration has been done. In that case, when taking the limit $z \to 0$, we obtain $lim_{z\to 0}I(z) = 0$. However, whenever an integration over any one of the $\{q_f, p_h\}$ variables at

a noise-pole is done, it has the effect of reducing a sum of the form $i(q+p)+2\epsilon$, $i(q+q')+2\epsilon$ or $i(p+p')+2\epsilon$ to zero in the denominator of $I(z)$. It means that the other member of the pair may be integrated around any of its poles without contributing to the denominator of $I(z)$, making that integral a possible non-zero contribution to $P^{ss}(x,v)$, when the limit $z \to 0$ is taken. Thus, we notice that only the residue integrations that eliminate the $\{q_f, p_h\}$ variables from the denominator of $I(z)$, two of them at a time, will reduce $I(z)$ to

$$lim_{z\to 0} I(z) = lim_{z\to 0}\frac{z}{z} = 1.$$

Only in this case, $I(z)=1$, the corresponding term will contribute to the stationary state distribution [26].

It is straightforward to show that cross terms that couple $\tilde{x}$ and $\tilde{v}$ ($\Omega\Theta$-type) do not contribute, while other terms coupling $\tilde{x}$ and $\tilde{x}$ ($\Omega\Omega$-type) or $\tilde{v}$ and $\tilde{v}$ ($\Theta\Theta$-type) will give non-zero contributions. As a consequence, the stationary probabilities for positions and velocities are independent: $P^{ss}(x,v)=P^{ss}(x)P^{ss}(v)$.

In the following, we evaluate some typical integrals and deduce the stationary distribution.

III. TIME-AVERAGING

A. Inertial case

In this section, we evaluate the terms that generate the stationary state distribution $P^{ss}(x,v)$ by analyzing the distinct contributions grouped into Eq.(27). So, lets start by studying a pair-integration contribution from a $\Omega\Omega-$correlation integral (for a position-position correlation this is a $[q,q']$ pair, such as mentioned in the last section):

$$\int_{-\infty}^{+\infty}\frac{dq_i}{2\pi}\frac{dq_j}{2\pi}\frac{z}{z-i(q_i+q_j-2i\epsilon)-i\diamond}\,\Omega(iq_i+\epsilon)\Omega(iq_j+\epsilon)\left[\langle\tilde{\eta}(iq_i+\epsilon)\tilde{\eta}(iq_j+\epsilon)\rangle+\langle\tilde{\xi}(iq_i+\epsilon)\tilde{\xi}(iq_j+\epsilon)\rangle\right]=\frac{z}{z-i\diamond}W_\Omega, \tag{29}$$

where we integrate over the contributing poles (only), and $i\diamond=\sum_{f=1,\neq i,j}^{l} iq_f+\sum_{h=1}^{m} ip_h+(l+m-2)\epsilon$ represents the summation over the p's and q's that are not integrated over in the denominator of $I(z)$. The W_Ω factor is given below (see Appendix C):

$$\begin{aligned} W_\Omega \;=\; & \frac{\Gamma_1 T_1}{2m^2}\,\frac{2z_1\mathcal{A}\mathcal{B}+(\mathcal{B}\mathcal{D}\mathcal{G}-\mathcal{B}\mathcal{C}\mathcal{F})+(\mathcal{A}\mathcal{D}\mathcal{F}+\mathcal{A}\mathcal{C}\mathcal{G})}{\mathcal{A}\mathcal{B}(\mathcal{C}^2+\mathcal{D}^2)(\mathcal{F}^2+\mathcal{G}^2)}+ \\ & -\frac{(\Gamma_1 T_1+\Gamma_2 T_2)}{2m^2\tau^2}\,\frac{2\mathcal{A}\mathcal{B}(\mathcal{A}^2+\mathcal{B}^2)-z_1(\mathcal{B}\mathcal{D}\mathcal{G}-\mathcal{B}\mathcal{C}\mathcal{F})+z_1(\mathcal{A}\mathcal{D}\mathcal{F}+\mathcal{A}\mathcal{C}\mathcal{G})}{z_1\mathcal{A}\mathcal{B}(\mathcal{A}^2+\mathcal{B}^2)(\mathcal{C}^2+\mathcal{D}^2)(\mathcal{F}^2+\mathcal{G}^2)}, \end{aligned} \tag{30}$$

The terms depending on $\Gamma_1 T_1$ ($\Gamma_2 T_2$) result from the integration over the poles of the $\langle\tilde{\eta}\tilde{\eta}'\rangle$ ($\langle\tilde{\xi}\tilde{\xi}'\rangle$), and the effect of the integration over q_i and q_j on Eq.(29) corresponds to the appearance of a W_Ω factor and the reduction of the denominator of $I(z)$ from $[z-i(q_i+q_j-2i\epsilon)-i\diamond]$ to $[z-i\diamond]$.

Similarly, a typical pair-integration contribution from a $\Theta\Theta-$correlation integral (velocity-velocity correlation $\to$ $[p,p']$ pair) reads:

$$\int_{-\infty}^{+\infty}\frac{dp_i}{2\pi}\frac{dp_j}{2\pi}\frac{z}{z-i(p_i+p_j-2i\epsilon)-i\diamond}\,\Theta(ip_i+\epsilon)\Theta(ip_j+\epsilon)\left[\langle\tilde{\eta}(ip_i+\epsilon)\tilde{\eta}(ip_j+\epsilon)\rangle+\langle\tilde{\xi}(ip_i+\epsilon)\tilde{\xi}(ip_j+\epsilon)\rangle\right]=\frac{z}{z-\diamond}W_\Theta \tag{31}$$

where the W_Θ factor is

$$\begin{aligned} W_\Theta \;=\; & \frac{(\Gamma_1 T_1+\Gamma_2 T_2)}{2m^2\tau^2}\,\frac{2z_1\mathcal{A}\mathcal{B}+(\mathcal{B}\mathcal{D}\mathcal{G}-\mathcal{B}\mathcal{C}\mathcal{F})+(\mathcal{A}\mathcal{D}\mathcal{F}+\mathcal{A}\mathcal{C}\mathcal{G})}{\mathcal{A}\mathcal{B}(\mathcal{C}^2+\mathcal{D}^2)(\mathcal{F}^2+\mathcal{G}^2)}+ \\ & -\frac{\Gamma_1 T_1}{2m^2}\,\frac{2z_1^3\mathcal{A}\mathcal{B}+(\mathcal{A}^3-3\mathcal{A}\mathcal{B}^2)(\mathcal{C}\mathcal{G}+\mathcal{D}\mathcal{F})+(\mathcal{B}^3-3\mathcal{A}^2\mathcal{B})(\mathcal{C}\mathcal{F}-\mathcal{D}\mathcal{G})}{\mathcal{A}\mathcal{B}(\mathcal{C}^2+\mathcal{D}^2)(\mathcal{F}^2+\mathcal{G}^2)}. \end{aligned} \tag{32}$$

It is straightforward to show that the integrals of the type $\Theta\Omega-$correlation (for a $[q,p]$ pair) over the correct poles will not contribute to $P^{ss}(x,v)$. It means that the information about the position distribution (q's) does not get mixed up with the information about the velocity distribution (p's). In other words, the distributions are independent ($P^{ss}(x,v)=P^{ss}(x)P^{ss}(v)$) as mentioned before.

Schematically, we can represent the integral in Eq.(27) by:

$$\lim_{z\to 0}\int \prod_{j=1}^{2n} dp_j \prod_{l=1}^{2m} dq_l \, \frac{z}{z-i\diamond} \prod_{ij}^{\text{all } p\text{-pairs}} \langle \tilde{\chi}_i \tilde{\chi}_j \rangle \prod_{kl}^{\text{all } q\text{-pairs}} \langle \tilde{\chi}_k \tilde{\chi}_l \rangle \to \begin{pmatrix} \text{all} \\ \text{terms} \end{pmatrix} W_\Theta^n \, W_\Omega^m ,$$

where $\tilde{\chi}$ represents the noise functions' Laplace transforms. The number of terms above remains to be evaluated.

First we notice that we can factor out the W_Θ^n contributions from the W_Ω^m contributions. By doing so, we only need to calculate the number of ways of integrating all $[q, q']$ pairs (and similarly for the $[p, p']$ pairs). Furthermore, averaging over a power $2t$ of $\tilde{\xi}$'s and $\tilde{\eta}$'s yields a number of terms analogous to the number of ways of distributing $2t$ balls into t boxes that have room for two balls each. The factor above is given by:

$$u_{2t} = \frac{(2t)!}{2^t \, t!}.$$

Let's replace that into the equation for $P^{ss}(x)$. The factored out contribution for the position distribution reads

$$P^{ss}(x) \;\to\; \sum_{l=0}^{\infty} \int_{-\infty}^{+\infty} \frac{dQ}{2\pi} e^{iQx} \frac{(-iQ)^{2l}}{(2l)!} \frac{(2l)!}{2^l \, l!} W_\Omega^l ,$$

$$\Rightarrow P^{ss}(x) \;=\; \frac{1}{\sqrt{2\,\pi\, W_\Omega}} e^{-\frac{x^2}{2\,W_\Omega}} . \tag{33}$$

Similarly, it is straightforward to show that:

$$P^{ss}(v) \;=\; \frac{1}{\sqrt{2\,\pi\, W_\Theta}} e^{-\frac{v^2}{2\,W_\Theta}} . \tag{34}$$

B. Stationary distribution

The Eqs.(33) and (34) combine to

$$P^{ss}(x,v) = P^{ss}(x) P^{ss}(v) = \frac{1}{2\,\pi\,\sqrt{W_\Theta \, W_\Omega}} \exp\left\{ -\frac{1}{2}\left(\frac{x^2}{W_\Omega} + \frac{v^2}{W_\Theta} \right) \right\} . \tag{35}$$

The distribution on Eq.(35) is complete, normalized and exact. It represents the stationary state for extremely long times and includes more information than similar models in the massless (over−damped) limit. Since $W_\Omega \neq W_\Theta$, a renormalization of the mass and of the temperature will occur. Recent works have suggested other mechanisms for deviations from Boltzmann statistics [27, 28, 29]. A massless BP is bound to move as the instantaneous force directs it, but an inertial one can accumulate kinetic energy and its velocity deviates from the ratio between the external force and the friction coefficient, creating an opportunity for a feed-back mechanism (incipiently present in the non-Markovian colored noise ξ time-scale τ) to modify the stationary distribution away from the Boltzmann equilibrium form [26, 30, 31, 32] even at $T_1 = T_2$ [26, 30]. Taking the limit $m \to 0$ at $T_1 = T_2$ restores the Boltzmann equilibrium form, despite $\tau > 0$, due to the crucial role of inertia in the mechanism above.

IV. OVER−DAMPED LIMIT

The Eq.(30) defining the W_Ω term, (see Appendix B for a complete derivation), gives the dependence of the stationary position distribution on all parameters of the system in its most general form. This distribution is a generalization of the expression obtained in Ref. [26] and, similarly to that case, it differs from the Boltzmann form due to the non-Markovian character of the colored noise, but tends to it when we take $\tau \to 0$ and $T_1 = T_2$. The nontrivial forms of Eqs.(30) and (32) indicate that a renormalization of the mass (or the rigidity k) takes place due to the non-Markovian character of ξ [26, 31, 32].

In the over damped limit, $m \to 0$, the term W_Ω simplifies considerably and reads

$$\lim_{m\to 0} W_\Omega = \frac{T_1\,\tau\,k + \Gamma_1\,T_1 + \Gamma_2\,T_2}{(\Gamma_1 + \Gamma_2 + k\tau)\,k} , \tag{36}$$

The quantity $k\,W_\Omega$ behaves as the exact effective temperature T_{eff} for the system in the over−damped limit. We observe that both the slow and the fast noise contribute to T_{eff}. It can be seen that $T_{eff} = k\,W_\Omega$ has an intermediate value between T_1 and T_2:

$$\min[T_1, T_2] \leq k\,W_\Omega \leq \max[T_1, T_2].$$

Comparing Eq.(36) with other approaches, we noticed that in Ref. [3] the authors show the steps to obtain the stationary distribution for a similar model in the over damped limit. In their case, they make approximations in such a way that both the slow field ξ and the dissipative memory term, proportional to Γ_2 (Γ_1 in their notation), behave as if they were effective external additive terms for the potential $V(x)$. In consequence, the frictional coefficient Γ_1 (Γ_0 in their notation) will not contribute to the equilibrium distribution since that is a function of the effective external terms only. The system is supposed to quickly adjust to that potential, giving rise to an instantaneous equilibrium distribution.

On the other hand, in the present method we calculate the averages rather differently. As we have seen, we first integrate over the noise-average of the exact distribution, Eq.(8), for an infinitely long-time and then obtain the time-average of them. By doing this, the neglected effect of Γ_1 becomes apparent and shows in our solution. So, in order to recover the results suggested on section 4.1 on Ref. [3] (derived in appendix C) all we need to do is take $\Gamma_1 = 0$ in Eq.(36). In our notation, the results on Ref. [3] read (where we assume $V(x) = \frac{k\,x^2}{2}$ so our models coincide)

$$W_\Omega^{[15]} = \frac{T_1\,\tau\,k + \Gamma_2\,T_2}{(\Gamma_2 + k\tau)\,k},$$

which is identical to ours when we take $\Gamma_1 = 0$ in Eq.(36).

When $T_1 = T_2 = T$ we recover the Boltzmann equilibrium with distribution

$$W_\Omega = \frac{T}{k} \Rightarrow P^{ss}(x) = \sqrt{\frac{k}{2\,\pi\,T}}\,e^{-\frac{kx^2}{2\,T}}. \tag{37}$$

For the velocity distribution, in the over−damped limit it assumes a simple form. As is the case of W_Ω, the expression for W_Θ in Eq.(32) expresses the dependence of the velocity distribution on all parameters of the model. However, by taking $m \to 0$ in Eq.(32) gives asymptotically:

$$\lim_{m\to 0} W_\Theta \;\to\; \frac{T_1}{m}. \tag{38}$$

We notice that only the fast noise T_1 contributes to the velocity distribution. The kinetic energy of the BP is completely driven by the fast noise and ignores the slow one in the over−damped limit.

The normalized velocity distribution reads in that approximation order:

$$W_\Theta = \frac{T_1}{m} \Rightarrow P^{ss}(v) = \sqrt{\frac{m}{2\,\pi\,T_1}}\,e^{-\frac{mv^2}{2\,T_1}}. \tag{39}$$

In this limit we observe that only the fast noise (T_1, Γ_1) drives the kinetic energy distribution.

V. CONCLUSIONS

A better understanding of slow dynamics and associated models may require powerful numerical methods, and computers, as one tries to simulate the (extremely) long-time relaxation that often occurs for interesting systems such as glasses.

In this context, simpler models may be very useful for obtaining some good qualitative understanding of the long time limit without presenting the numerical difficulties the more realistic ones do, with the possibility of obtaining exact results. These are always interesting in that they take into account all the physical effects present in a given model, independently of any approximation. In other words, exact treatments contain all the available information about a model. Some of it being inaccessible through approximate methods.

In that spirit, BP models have been proposed as a means to study some of the phenomena associated with the competition between fast thermal fluctuations and slow structural relaxation in glasses [3, 23, 24]. A white noise (fast) is associated with the thermal fluctuations that happen in short time-scales. A colored noise function (slow) is associated with the long time relaxation (structural).

For these simple models, a typical approach is to consider the over-damped regime, which is equivalent to the case when the BP has zero mass ($m \to 0^+$). When responding to external forces acting on it the BP's velocity assumes the ratio between the sum of all external forces and a friction coefficient. These models can be used to study the emergence of a non-equilibrium stationary state as the probability distribution tends to the stationary form as $t \to \infty$.

In the present work, we develop an exact method for time-averaging the distribution for x and v for a BP submitted to a fast white noise and a slow non-Markovian one at different temperatures.

The essence of our method is the use of time-averages of the distributions for x and v which leads us naturally to the use of Laplace transformations in order to "open" the problem. In a straightforward way, we represent the time-averaged distribution as a sum of integrals that can be easily analyzed, and computed exactly. The distribution thus obtained can be compared to the ones in the literature. By being exact, our method allows us to obtain information about the effects due to the finite mass of the BP showing, for instance, that the fast noise term plays a role on the stationary distribution of positions that is usually neglected in other methods.

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APPENDIX A

The Laplace transformation of Eq.(27) is shown below [26]. We will give a step by step derivation for the expression of $P^{ss}(x,v)$ as a function of the Laplace transforms of the position and velocity for the Brownian particle.

Lets start with the basic definition:

$$P^{ss}(x,v) \;=\; \lim_{z\to 0^+} z\int_0^\infty e^{-zt}\langle\delta(x-x(t))\delta(v-v(t))\rangle dt.$$

We write the delta-functions above in the integral representation:

$$\langle\delta(x-x(t))\delta(v-v(t))\rangle \;=\; \int_{-\infty}^{+\infty}\frac{dQ}{2\pi}e^{iQx}\int_{-\infty}^{+\infty}\frac{dP}{2\pi}e^{iPv}\sum_{l=0}^{\infty}\frac{(-iQ)^l}{l!}\sum_{m=0}^{\infty}\frac{(-iP)^m}{m!}\langle x^l(t)v^m(t)\rangle,$$

and obtain (after using the delta functions to express identically the averages over the noise as functions at distinct times):

$$\begin{aligned} P^{ss}(x,v) \;=\; & \lim_{z\to 0^+} z\int_0^\infty dt e^{-zt}\int_{-\infty}^{+\infty}\frac{dQ}{2\pi}e^{iQx}\int_{-\infty}^{+\infty}\frac{dP}{2\pi}e^{iPv}\sum_{l=0}^{\infty}\frac{(-iQ)^l}{l!}\sum_{m=0}^{\infty}\frac{(-iP)^m}{m!} \\ & \times\int_0^\infty\prod_{f=1}^{l}dt_{lf}\delta(t-t_{lf})\int_0^\infty\prod_{h=1}^{m}dt_{mh}\delta(t-t_{mb})\langle\prod_{f=1}^{l}x(t_{lf})\prod_{h=1}^{m}v(t_{mh})\rangle. \end{aligned}$$

Next, we express all delta functions above as integrals on the complex plane, displaced from the complex axis by a factor of ϵ (that vanishes faster than z). That factor will guarantee the convergence of the Laplace transforms for positions and velocities in the following.

$$\begin{aligned} P^{ss}(x,v) \;=\; & \lim_{z,\epsilon\to 0^+} z\int_0^\infty dt e^{-zt}\int_{-\infty}^{+\infty}\frac{dQ}{2\pi}e^{iQx}\int_{-\infty}^{+\infty}\frac{dP}{2\pi}e^{iPv}\sum_{l=0}^{\infty}\frac{(-iQ)^l}{l!}\sum_{m=0}^{\infty}\frac{(-iP)^m}{m!} \\ & \times\int_{-\infty}^{+\infty}\prod_{f=1}^{l}\frac{dq_f}{2\pi}\prod_{h=1}^{m}\frac{dp_h}{2\pi}\int_0^\infty\prod_{f=1}^{l}dt_{lf}\int_0^\infty\prod_{h=1}^{m}dt_{mh} \\ & \times\; e^{\sum_{f=1}^{l}(t-t_{lf})(iq_f+\epsilon)+\sum_{h=1}^{m}(t-t_{mh})(ip_h+\epsilon)}\langle\prod_{f=1}^{l}x(t_{lf})\prod_{h=1}^{m}v(t_{mh})\rangle. \end{aligned}$$

We need to integrate over all $\{t_{lf},t_{mh}\}$, obtaining the averages over the Laplace transforms of the position and velocity:

$$\begin{aligned} P^{ss}(x,v) \;=\; & \lim_{z,\epsilon\to 0^+}\int_{-\infty}^{+\infty}\frac{dQ}{2\pi}e^{iQx}\int_{-\infty}^{+\infty}\frac{dP}{2\pi}e^{iPv}\sum_{l=0}^{\infty}\frac{(-iQ)^l}{l!}\sum_{m=0}^{\infty}\frac{(-iP)^m}{m!}\int_{-\infty}^{+\infty}\prod_{f=1}^{l}\frac{dq_f}{2\pi}\prod_{h=1}^{m}\frac{dp_h}{2\pi} \\ & \times\int_0^\infty dt\, z\, e^{-t\left\{z-\sum_{f=1}^{l}(iq_f+\epsilon)-\sum_{h=1}^{m}(ip_h+\epsilon)\right\}}\langle\prod_{f=1}^{l}\tilde{x}(iq_f+\epsilon)\prod_{h=1}^{m}\tilde{v}(ip_h+\epsilon)\rangle. \end{aligned}$$

Finally, we integrate over t and obtain Eq.(27):

$$\begin{aligned} P^{ss}(x,v) \;=\; & \lim_{z,\epsilon\to 0^+}\sum_{l,m=0}^{\infty}\int_{-\infty}^{+\infty}\frac{dQ}{2\pi}\frac{dP}{2\pi}e^{iQx+iPv}\frac{(-iQ)^l}{l!}\frac{(-iP)^m}{m!}\int_{-\infty}^{+\infty}\prod_{f=1}^{l}\frac{dq_f}{2\pi}\prod_{h=1}^{m}\frac{dp_h}{2\pi} \\ & \times\;\frac{z}{z-\left[\sum_{f=1}^{l}iq_f+\sum_{h=1}^{m}ip_h+(l+m)\epsilon\right]}\langle\prod_{f=1}^{l}\tilde{x}(iq_f+\epsilon)\prod_{h=1}^{m}\tilde{v}(ip_h+\epsilon)\rangle \end{aligned} \tag{A1}$$

APPENDIX B

A typical calculation of the stationary state distribution of displacements terms is shown below for the W_Ω term from Eq.(29).

The first integral in Eq.(29) reads:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi}\frac{z}{z-i(q_i+q_j-2i\,\epsilon+\diamond)}\,\Omega(iq_i+\epsilon)\Omega(iq_j+\epsilon)\langle\tilde\eta(iq_i+\epsilon)\tilde\eta(iq_j+\epsilon)\rangle = \\
&= \int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi}\frac{z}{z-i(q_i+q_j-2i\,\epsilon+\diamond)}\,\frac{1+\tau\,(iq_i+\epsilon)}{[q_i-i(\epsilon-z_1)]\,[q_i-i(\epsilon-z_2)]\,[q_i-i(\epsilon-z_3)]} \\
&\times \frac{1+\tau\,(iq_i+\epsilon)}{[q_j-i(\epsilon-z_1)]\,[q_j-i(\epsilon-z_2)]\,[q_j-i(\epsilon-z_3)]}\,\frac{\Gamma_1\,T_1\,m^{-2}\,\tau^{-2}}{(-i)[q_j-(-q_i+2\,i\,\epsilon)]}
\end{aligned}$$

To continue the calculation, the integrations over the poles must be done in a way that allows us to obtain, after all integrations have been done (as explained in Section III.A), $\lim_{z\to 0} I(z) = 1$. When integrating over the poles of q_j's it is possible to see, Fig.(1), that it will work only for $q_j = -q_i + 2\,i\,\epsilon$. Thus:

$$\begin{aligned}
&\Rightarrow \int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi}\frac{z}{z-i(q_i+q_j-2i\,\epsilon+\diamond)}\,\Omega(iq_i+\epsilon)\Omega(iq_j+\epsilon)\langle\tilde\eta(iq_i+\epsilon)\tilde\eta(iq_j+\epsilon)\rangle = \\
&= \frac{\Gamma_1\,T_1}{m^2\,\tau^2}\,\frac{z}{z-i\diamond}\int_{-\infty}^{+\infty}\frac{dq_i}{2\pi}\frac{1-\tau^2\,(iq_i+\epsilon)^2}{[q_i-i(\epsilon-z_1)]\,[q_i-i(\epsilon-z_2)]\,[q_i-i(\epsilon-z_3)]\,[q_i-i(\epsilon+z_1)]\,[q_i-i(\epsilon+z_2)]\,[q_i-i(\epsilon+z_3)]}
\end{aligned}$$

The same holds for the integration over the q_i's poles. So, the non-zero contribution comes only from the poles $q_i = i(\epsilon - z_\alpha), \alpha = 1, 2, 3$, in the upper part of Fig.(1):

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi}\frac{z}{z-i(q_i+q_j-2i\,\epsilon+\diamond)}\,\Omega(iq_i+\epsilon)\Omega(iq_j+\epsilon)\langle\tilde\eta(iq_i+\epsilon)\tilde\eta(iq_j+\epsilon)\rangle = \\
&= \frac{\Gamma_1\,T_1}{m^2\,\tau^2}\,\frac{z}{z-i\diamond}\left\{\frac{\tau^2\,z_1^2-1}{z_1\,\mid z_1-z_2\mid^2\mid z_1+z_2\mid^2} - \frac{\tau^2[z_2\,(z_1-z_2)^*(z_1+z_2)^* - z_2^*\,(z_1-z_2)(z_1+z_2)]}{4\,i\,\Re(z_2)\,\Im(z_2)\mid z_1-z_2\mid^2\mid z_1+z_2\mid^2}\right. \\
&+ \left.\frac{[z_2^*(z_1-z_2)^*(z_1+z_2)^* - z_2(z_1-z_2)(z_1+z_2)]}{4\,i\,\Re(z_2)\,\Im(z_2)\mid z_2\mid^2\mid z_1-z_2\mid^2\mid z_1+z_2\mid^2}\right\}
\end{aligned}$$

Finally, using the definitions of Eq.(24), we get for the first part of Eq.(29):

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi}\frac{z}{z-i(q_i+q_j-2i\,\epsilon+\diamond)}\,\Omega(iq_i+\epsilon)\Omega(iq_j+\epsilon)\langle\tilde\eta(iq_i+\epsilon)\tilde\eta(iq_j+\epsilon)\rangle = \\
&= \frac{z}{z-i\diamond}\left\{\frac{\Gamma_1\,T_1}{2\,m^2}\,\frac{2\,z_1\,\mathcal{A}\mathcal{B}+(\mathcal{B}\mathcal{D}\mathcal{G}-\mathcal{B}\mathcal{C}\mathcal{F})+(\mathcal{A}\mathcal{D}\mathcal{F}+\mathcal{A}\mathcal{C}\mathcal{G})}{\mathcal{A}\mathcal{B}\,(\mathcal{C}^2+\mathcal{D}^2)\,(\mathcal{F}^2+\mathcal{G}^2)}\right. \\
&- \left.\frac{\Gamma_1\,T_1}{2\,m^2\,\tau^2}\,\frac{2\,\mathcal{A}\mathcal{B}\,(\mathcal{A}^2+\mathcal{B}^2)-z_1\,(\mathcal{B}\mathcal{D}\mathcal{G}-\mathcal{B}\mathcal{C}\mathcal{F})+z_1\,(\mathcal{A}\mathcal{D}\mathcal{F}+\mathcal{A}\mathcal{C}\mathcal{G})}{z_1\,\mathcal{A}\mathcal{B}\,(\mathcal{A}^2+\mathcal{B}^2)\,(\mathcal{C}^2+\mathcal{D}^2)\,(\mathcal{F}^2+\mathcal{G}^2)}\right\}
\end{aligned}$$

For the second part of Eq.(29), the second term of the RHS of Eq.(11) does not contribute since it is straightforward to show that it leads to a null contribution. Thus, we shall compute the contribution from the first part of Eq.(29) keeping only the RHS of Eq.(11). It yields:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi}\frac{z}{z-i(q_i+q_j-2i\,\epsilon+\diamond)}\,\Omega(iq_i+\epsilon)\Omega(iq_j+\epsilon)\langle\tilde\xi(iq_i+\epsilon)\tilde\xi(iq_j+\epsilon)\rangle = \\
&= \int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi}\frac{z}{z-i(q_i+q_j-2i\,\epsilon+\diamond)}\,\frac{1+\tau\,(iq_i+\epsilon)}{[q_i-i(\epsilon-z_1)]\,[q_i-i(\epsilon-z_2)]\,[q_i-i(\epsilon-z_3)]} \\
&\times \frac{1+\tau\,(iq_i+\epsilon)}{[q_j-i(\epsilon-z_1)]\,[q_j-i(\epsilon-z_2)]\,[q_j-i(\epsilon-z_3)]}\,\frac{\Gamma_2\,T_2\,m^{-2}\,\tau^{-2}}{(-i)[q_j-(-q_i+2\,i\,\epsilon)][1-\tau(iq_i+\epsilon)][1-\tau(iq_j+\epsilon)]}
\end{aligned} \tag{B1}$$

The only contributing integrations will be the ones over the poles $q_j = -q_i + 2\,i\,\epsilon$, (see Fig.(1)). So:

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi}\frac{z}{z-i(q_i+q_j-2i\,\epsilon+\diamond)}\,\Omega(iq_i+\epsilon)\Omega(iq_j+\epsilon)\langle\tilde{\xi}(iq_i+\epsilon)\tilde{\xi}(iq_j+\epsilon)\rangle =$$
$$= \frac{\Gamma_2\,T_2}{m^2\,\tau^2}\,\frac{z}{z-i\diamond}\int_{-\infty}^{+\infty}\frac{dq_i}{2\pi}\frac{1}{[q_i-i(\epsilon-z_1)]\,[q_i-i(\epsilon-z_2)]\,[q_i-i(\epsilon-z_3)]}\frac{1}{[q_i-i(\epsilon+z_1)]\,[q_i+i(\epsilon-z_2)]\,[q_i+i(\epsilon-z_3)]}$$

Again, the contribution came from the poles $q_i = i(\epsilon - z_\alpha), \alpha = 1, 2, 3$. So:

$$\int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi}\frac{z}{z-i(q_i+q_j-2i\,\epsilon+\diamond)}\,\Omega(iq_i+\epsilon)\Omega(iq_j+\epsilon)\langle\tilde{\xi}(iq_i+\epsilon)\tilde{\xi}(iq_j+\epsilon)\rangle =$$
$$= \frac{\Gamma_2\,T_2}{2\,m^2\,\tau^2}\,\frac{z}{z-i\diamond}\left\{\frac{-1}{z_1\;|\;z_1-z_2\;|^2\;|\;z_1+z_2\;|^2}+\frac{[z_2^*(z_1-z_2)^*(z_1+z_2)^*-z_2(z_1-z_2)(z_1+z_2)]}{4\,i\,\Re(z_2)\,\Im(z_2)\;|\;z_2\;|^2|\;z_1-z_2\;|^2\;|\;z_1+z_2\;|^2}\right\}.$$

Using the definitions on Eq.(24) we obtain:

$$\int_{-\infty}^{+\infty} \frac{dq_i}{2\pi}\frac{dq_j}{2\pi}\frac{z}{z-i(q_i+q_j-2i\,\epsilon+\diamond)}\,\Omega(iq_i+\epsilon)\Omega(iq_j+\epsilon)\langle\tilde{\xi}(iq_i+\epsilon)\tilde{\xi}(iq_j+\epsilon)\rangle =$$
$$= -\frac{\Gamma_2\,T_2}{2\,m^2\,\tau^2}\,\frac{2\,\mathcal{A}\,\mathcal{B}\,(\mathcal{A}^2+\mathcal{B}^2)-z_1\,(\mathcal{B}\,\mathcal{D}\,\mathcal{G}-\mathcal{B}\,\mathcal{C}\,\mathcal{F})+z_1\,(\mathcal{A}\,\mathcal{D}\,\mathcal{F}+\mathcal{A}\,\mathcal{C}\,\mathcal{G})}{z_1\,\mathcal{A}\,\mathcal{B}\,(\mathcal{A}^2+\mathcal{B}^2)\,(\mathcal{C}^2+\mathcal{D}^2)\,(\mathcal{F}^2+\mathcal{G}^2)} \tag{B2}$$

Combining Eqs.(B1) and (B2) we obtain the result for W_Ω in Eq.(30). An analogous calculation can be done for the velocity integrals and obtain W_Θ, e.g., Eq.(32).

APPENDIX C

The stationary distribution of x is obtained in Cugliandolo and Kurchan [3]. We review their main steps below. Let's define:

$$P(x) = \int dh P(x/h) P(h), \tag{C1}$$

where

$$P(h) = \frac{e^{-\beta^*(F(h)+\frac{h^2}{2\frac{\Gamma_2}{\tau}})}}{\int dh\, e^{-\beta^*(F(h)+\frac{h^2}{2\frac{\Gamma_2}{\tau}})}}, \tag{C2}$$

and

$$P(x/h) = \frac{e^{-\beta(V(x)+\frac{\Gamma_2}{\tau}\frac{x^2}{2}-h\,x)}}{\int dx\, e^{-\beta(V(x)+\frac{\Gamma_2}{\tau}\frac{x^2}{2}-h\,x)}}. \tag{C3}$$

The denominator above defines $Z(h)$ and $F(h) \equiv -\beta^{-1} \ln Z(h)$. For $V(x) = k\frac{x^2}{2}$, we have:

$$Z(h) = \sqrt{\frac{2\,\pi}{\beta\,(k+\frac{\Gamma_2}{\tau})}}\, e^{\frac{\beta\,h^2}{2\,(k+\frac{\Gamma_2}{\tau})}};\ F(h) = -\frac{h^2}{2\,(k+\frac{\Gamma_2}{\tau})} - \beta^{-1}C \tag{C4}$$

And the denominator of Eq.(C2) becomes:

$$\int dh\, e^{-\beta^*\left(-\frac{h^2}{2\,(k+\frac{\Gamma_2}{\tau})}-\beta^{-1}C+\frac{h^2}{2\frac{\Gamma_2}{\tau}}\right)} = e^{\frac{\beta^*\,C}{\beta}}\sqrt{\frac{2\,\pi\,\frac{\Gamma_2}{\tau}\,(k+\frac{\Gamma_2}{\tau})}{\beta^*\,k}} \tag{C5}$$

Thus, Eq.(C1) becomes:

$$P(x) = \sqrt{\frac{\beta^*\,\beta\;k\,(k\tau+\Gamma_2)}{2\,\pi\,(\beta^*\,k\tau+\beta\,\Gamma_2)}}\, e^{-\frac{\beta^*\,\beta\;k\,(k\tau+\Gamma_2)\,x^2}{2\,(\beta^*\,k\tau+\beta\,\Gamma_2)}} \tag{C6}$$

When the temperatures are equal, then $\beta^* = \beta = 1/T$ and:

$$P(x) = \sqrt{\frac{k}{2\,\pi\,T}}\, e^{-\frac{k\,x^2}{2\,T}} \tag{C7}$$

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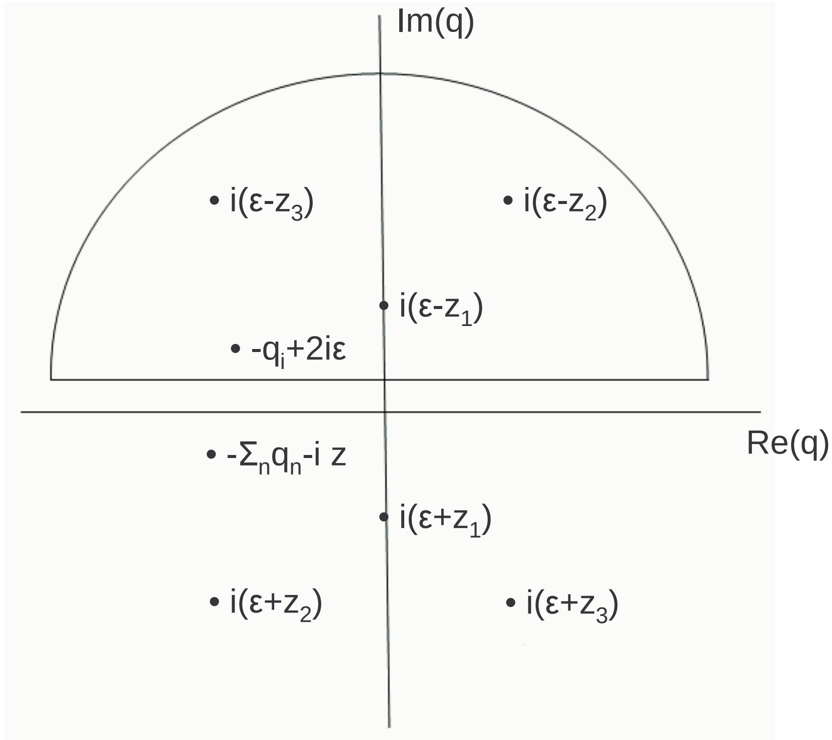

FIG. 1: Integration path for the q or p–variables.